Basics on probability theory and reliability design

Typical probability distribution function 1

- Normal distribution (Gaussian dist.)
  - Most widely used to represent variation of values in engineering field.
  - Parameters in the function are only the average and standard deviation.

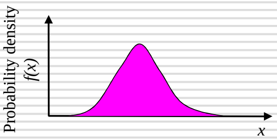
  Expamples:

  Distribution of
  - ✓ number of appearance of even numbers when playing a dice N times
  - ✓ height of people
  - ✓ scores in an examination
  - ✓ compressive strength of concrete

Typical probability distribution function 2

- Probability distribution function for Normal Distribution

$$f(x)=\frac{1}{\sqrt{2\pi}\sigma_x}e^{-\frac{1}{2}\left(\frac{x-\bar{x}}{\sigma_x}\right)^2}$$



σ_x : Standard deviation

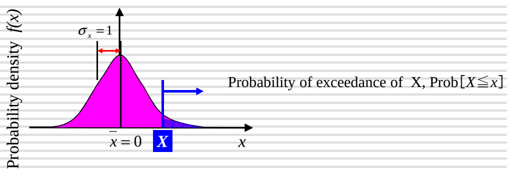
average $\bar{x}=E[x]=\int_{-\infty}^{\infty} x f(x)dx$

$$\sigma_x^2=E\left[(x-\bar{x})^2\right]=\int_{-\infty}^{\infty}(x-\bar{x})^2 f(x)dx$$
$$=E[x^2]-2\bar{x}E[x]+\bar{x}^2$$
$$=E[x^2]-\bar{x}^2$$

Variation coefficient $V_x=\frac{\sigma_x}{\bar{x}}$

Typical probability distribution function 3

- Standard normal distribution
  - Normal distribution with average=0 and σ_x=1

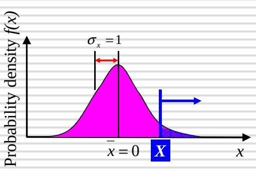


X	1.0	1.5	2.0	2.5	3.0	4.0
Prob[X≦x] (%)	15.8	6.7	2.3	0.62	0.14	0.003

Typical probability distribution function 4

- Log-normal distribution
  - Natural logarithm of X, ln x is compatible with a normal distribution.
  - Widely applied to a distribution of variables X with positive values only.

$$f(x)=\frac{1}{\sqrt{2\pi}\sigma_{\ln x}}e^{-\frac{1}{2}\left(\frac{\ln x-\overline{\ln x}}{\sigma_{\ln x}}\right)^2}$$



question 1

- Assume heights of boy students in Tohoku Univ. are compatible with a normal distribution with an average of 170cm and variation coefficient of 10%.

  Evaluate the percentage of students of 2m or taller

Probabilistic and deterministic theory 1

- Current structural design
  - "Stress S in a structure must not exceed resistance R". ➔ deterministic approach

  However,

  A structure which never fail due to huge earthquake does not exist.
  - Building code assure a safety only against specified design load.
  - How Safe against loads larger than specifications?
  - How much is the margin to design load?

Probabilistic and deterministic theory 2

- Reliability design
  - "probability of exceedance of stress S to resistance R is 5%".

    ➔ probabilistic approach
- Limit state design

  AIJ, "Guidelines for limit state design of buildings", 2004.

  probability of occurrence of limit state is designed lower than allowable value.

  "limit state" is a state in which a structure keep its performance required in the design.

  ✓ Serviceability, repairability, safety limit state

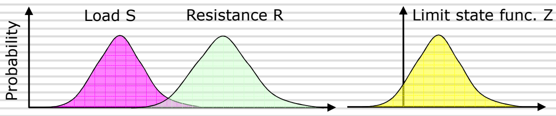
Limit state function and probability of failure

- Limit state function is defined by random variables, load S and resistance R.

Z=R-S

When S>R, structure fails.

i.e., probability of Z≦0 is failure probability, Pf

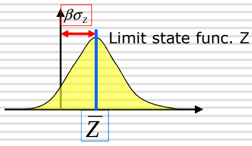


Limit state func. Z and reliability index β

- When both of load S and resistance R are probability variables, limit state function is also probability variable.

$$\overline{Z}=\overline{R}-\overline{S}$$

$$\sigma_Z=\sqrt{\sigma_R^{\ 2}+\sigma_S^{\ 2}}$$



- Reliability index β

  Failure probability Pf is obtained by Z function.

$$\beta=\frac{\overline{Z}}{\sigma_Z}=\frac{\overline{R}-\overline{S}}{\sqrt{\sigma_R^{\ 2}+\sigma_S^{\ 2}}}$$

Ref.

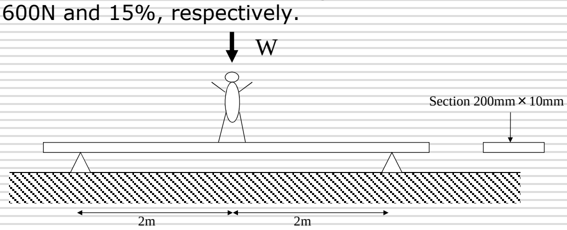
Average and standard deviation of difference Z between independent variables X and Y

$$\overline{Z}=\overline{X-Y}=\frac{\sum_{n=1}^{N}(X_i-Y_i)}{N}=\frac{\sum_{n=1}^{N}X_i}{N}-\frac{\sum_{n=1}^{N}Y_i}{N}=\overline{X}-\overline{Y}$$

$$\begin{aligned}\sigma_z^{\ 2}&=\overline{Z^2}-\overline{Z}^2=\overline{(X-Y)^2}-(\overline{X}-\overline{Y})^2\\&=\left(\overline{X^2}-2\overline{XY}+\overline{Y^2}\right)-\left(\overline{X}^2-2\overline{X}\cdot\overline{Y}+\overline{Y}^2\right)\\&=\left(\overline{X^2}-\overline{X}^2\right)+\left(\overline{Y^2}-\overline{Y}^2\right)+\left(2\overline{XY}-2\overline{X}\cdot\overline{Y}\right)\\&=\sigma_X^{\ 2}+\sigma_Y^{\ 2}\end{aligned}$$

Question 2

a man is standing at the center of a simply supported beam shown in the figure below. The beam is made by steel with a 200mm by 10mm section and 4m of span. Young's modulus is 2×10^5 N/mm^2. The average and variation coefficient of weight of the man are 600N and 15%, respectively.



Question 2 (cont'd)

1. Height from the floor to the beam is 0.3m deterministically. Find reliability index β for landing of the beam due to the load.
2. Assume beam is high enough above the floor. Yielding strength, σ_Y of steel is 300 N/mm^2 in an average with variation coefficient of 10%. Find reliability index β for yielding of the.

Three levels of limit state design

- Level 3: evaluate probability of failure based on accurate distribution of all the uncertain parameters.
- Level 2: evaluate reliability index β based on the average and standard deviation of limit state function, Z. β is larger than target β_T.
  (normal distribution is assumed.)
- Level 1: evaluate partial safety factors. Loads and resistances are amplified/reduced by the factors.
  verify (load)≦(resistance) in design.
  Procedure of design is similar with conventional calculation, so that relatively easy to understand to practitioners and engineers.
  (partial safety factors methods)

Load and resistance factors

$\beta = \frac{\bar{Z}}{\sigma_Z} = \frac{\bar{R}-\bar{S}}{\sqrt{\sigma_R^{\,2}+\sigma_S^{\,2}}} \geq \beta_T$ can be expressed by $\bar{R} \geq \bar{S} + \beta_T\sqrt{\sigma_R^{\,2}+\sigma_S^{\,2}}$

with separation factors α_R, α_S, approximation by $\sqrt{\sigma_R^{\,2}+\sigma_S^{\,2}} \approx \alpha_R\sigma_R + \alpha_S\sigma_S$

$$\bar{R} - \beta_T\alpha_R\sigma_R \geq \bar{S} + \beta_T\alpha_S\sigma_S$$

$$\left(1-\beta_T\alpha_R\frac{\sigma_R}{\bar{R}}\right)\bar{R} \geq \left(1+\beta_T\alpha_S\frac{\sigma_S}{\bar{S}}\right)\bar{S}$$

$$(1-\beta_T\alpha_R V_R)\bar{R} \geq (1+\beta_T\alpha_S V_S)\bar{S}$$

Below is a variation of expression with characteristic value of load, S_n and nominal strength R_n

$$(1-\beta_T\alpha_R V_R)\frac{\bar{R}}{R_n}R_n \geq (1+\beta_T\alpha_S V_S)\frac{\bar{S}}{S_n}S_n$$

Resistance factor $\phi = (1-\beta_T\alpha_R V_R)\frac{\bar{R}}{R_n}$

Load factor $\gamma = (1+\beta_T\alpha_S V_S)\frac{\bar{S}}{S_n}$

Separation factor $\alpha_R = \frac{\sigma_R}{\sqrt{\sigma_R^{\,2}+\sigma_S^{\,2}}} \approx \frac{V_R}{\sqrt{V_R^{\,2}+V_S^{\,2}}}$

$\alpha_S = \frac{\sigma_S}{\sqrt{\sigma_R^{\,2}+\sigma_S^{\,2}}} \approx \frac{V_S}{\sqrt{V_R^{\,2}+V_S^{\,2}}}$

Partial safety factor method

(resistance factor)×(resistance) ≧
(load factor)×(load)

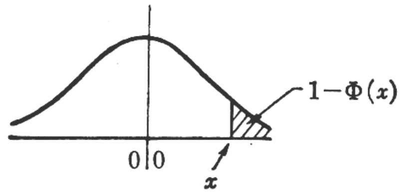
$$\phi R_n \geq \gamma S_n$$

- references

  - Jun kanda, "Recommendation of Limit State Design", Kenchikugijyutu, 1993.
  - Masaru Hoshiya and Kiyoshi Ishii, "Reliability design of structures", kajima Institute Publishing, 1986.

Area for standard normal probability density function

$$1-\Phi(x)=\int_x^{\infty}\frac{1}{\sqrt{2\pi}}e^{-x^2/2}dx$$



x	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	.5000	.4960	.4920	.4880	.4840	.4801	.4761	.4721	.4681	.4641
0.1	.4602	.4562	.4522	.4483	.4443	.4404	.4364	.4325	.4286	.4247
0.2	.4207	.4168	.4129	.4090	.4052	.4013	.3974	.3936	.3897	.3859
0.3	.3821	.3783	.3745	.3707	.3669	.3632	.3594	.3557	.3520	.3483
0.4	.3446	.3409	.3372	.3336	.3300	.3264	.3228	.3192	.3156	.3121
0.5	.3085	.3050	.3015	.2981	.2946	.2912	.2877	.2843	.2810	.2776
0.6	.2743	.2709	.2676	.2643	.2611	.2578	.2546	.2514	.2483	.2451
0.7	.2420	.2389	.2358	.2327	.2296	.2266	.2236	.2206	.2177	.2148
0.8	.2119	.2090	.2061	.2033	.2005	.1977	.1949	.1922	.1894	.1867
0.9	.1841	.1814	.1788	.1762	.1736	.1711	.1685	.1660	.1635	.1611
1.0	.1587	.1562	.1539	.1515	.1492	.1469	.1446	.1423	.1401	.1379
1.1	.1357	.1335	.1314	.1292	.1271	.1251	.1230	.1210	.1190	.1170
1.2	.1151	.1131	.1112	.1093	.1075	.1056	.1038	.1020	.1003	.0985
1.3	.0968	.0951	.0934	.0918	.0901	.0885	.0869	.0853	.0838	.0823
1.4	.0808	.0793	.0778	.0764	.0749	.0735	.0721	.0708	.0694	.0681
1.5	.0668	.0655	.0643	.0630	.0618	.0606	.0594	.0582	.0571	.0559
1.6	.0548	.0537	.0526	.0516	.0505	.0495	.0485	.0475	.0465	.0455
1.7	.0446	.0436	.0427	.0418	.0409	.0401	.0392	.0384	.0375	.0367
1.8	.0359	.0351	.0344	.0336	.0329	.0322	.0314	.0307	.0301	.0294
1.9	.0287	.0281	.0274	.0268	.0262	.0256	.0250	.0244	.0239	.0233
2.0	.0228	.0222	.0217	.0212	.0207	.0202	.0197	.0192	.0188	.0183
2.1	.0179	.0174	.0170	.0166	.0162	.0158	.0154	.0150	.0146	.0143
2.2	.0139	.0136	.0132	.0129	.0125	.0122	.0119	.0116	.0113	.0110
2.3	.0107	.0104	.0102	.00990	.00964	.00939	.00914	.00889	.00866	.00842
2.4	.00820	.00798	.00776	.00755	.00734	.00714	.00695	.00676	.00657	.00639
2.5	.00621	.00604	.00587	.00570	.00554	.00539	.00523	.00508	.00494	.00480
2.6	.00466	.00453	.00440	.00427	.00415	.00402	.00391	.00379	.00368	.00357
2.7	.00347	.00336	.00326	.00317	.00307	.00298	.00289	.00280	.00272	.00264
2.8	.00256	.00248	.00240	.00233	.00226	.00219	.00212	.00205	.00199	.00193
2.9	.00187	.00181	.00175	.00169	.00164	.00159	.00154	.00149	.00144	.00139

x	.0	.1	.2	.3	.4	.5	.6	.7	.8	.9
3	.00135	$.0^{3}968$	$.0^{3}687$	$.0^{3}483$	$.0^{3}337$	$.0^{3}233$	$.0^{3}159$	$.0^{3}108$	$.0^{4}723$	$.0^{4}481$
4	$.0^{4}317$	$.0^{4}207$	$.0^{4}133$	$.0^{5}854$	$.0^{5}541$	$.0^{5}340$	$.0^{5}211$	$.0^{5}130$	$.0^{6}793$	$.0^{6}479$
5	$.0^{6}287$	$.0^{6}170$	$.0^{7}996$	$.0^{7}579$	$.0^{7}333$	$.0^{7}190$	$.0^{7}107$	$.0^{8}599$	$.0^{8}332$	$.0^{8}182$
6	$.0^{9}987$	$.0^{9}530$	$.0^{9}282$	$.0^{9}149$	$.0^{10}777$	$.0^{10}402$	$.0^{10}206$	$.0^{10}104$	$.0^{11}523$	$.0^{11}260$